United Kingdom
Mathematics Trust

Senior Mathematical Challenge

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Solutions and investigations

7 November 2019

These solutions augment the shorter solutions also available online. For convenience, the shorter solutions are confined to four pages and therefore in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	B	A	B	B	D	D	B	D	C	C	E	D	C	B	A	E	E	A	A	D	C	B	E	A

1. What is the value of $123^2 - 23^2$?

A 10 000 B 10 409 C 12 323 D 14 600 E 15 658

SOLUTION **D**

COMMENTARY

The SMC is a *no calculator* paper. This should be a clue that there is a better way to answer this question than separately squaring 123 and 23 and then doing a subtraction.

There is no great virtue in being able to evaluate $123^2 - 23^2$ without a calculator, but the underlying method of factorizing the difference of two squares $x^2 - y^2$ as $(x - y)(x + y)$ is very useful and should be remembered.

We evaluate $123^2 - 23^2$ as follows.

$$\begin{aligned} 123^2 - 23^2 &= (123 - 23)(123 + 23) \\ &= 100 \times 146 \\ &= 14\,600. \end{aligned}$$

FOR INVESTIGATION

1.1 Find the values of

(a) $57^2 - 43^2$,

(b) $203^2 - 197^2$,

(c) $2019^2 - 2018^2$.

1.2 Find all the pairs (a, b) of positive integers such that

$$a^2 = b^2 + 2019.$$

1.3 Factorize

$$x^4 - y^4.$$

1.4 Prove that if p and q are both prime numbers, with $p > q > 2$, then $p^4 - q^4$ is divisible by 16.

1.5 Use the fact [see Exercise 10.2] that each prime greater than 3 has the form $6n \pm 1$ for some integer n, to prove that if p and q are primes with $p > q > 3$, then $p^4 - q^4$ is divisible by 48.

1.6 Prove that if p and q are both prime numbers, with $p > q > 5$, then $p^4 - q^4$ is divisible by 240.

2. What is the value of $(2019 - (2000 - (10 - 9))) - (2000 - (10 - (9 - 2019)))$?

A 4040 B 40 C −400 D −4002 E −4020

SOLUTION **B**

We evaluate the expression by first evaluating the innermost brackets and then working outwards.

$$\begin{aligned}(2019 - (2000 - (10 - 9))) - (2000-(10 - (9 - 2019)))\\ &= (2019 - (2000 - 1)) - (2000 - (10 - (-2010)))\\ &= (2019 - (2000 - 1)) - (2000 - (10 + 2010))\\ &= (2019 - 1999) - (2000 - 2020)\\ &= (2019 - 1999) - (-20)\\ &= 20 + 20\\ &= 40.\end{aligned}$$

FOR INVESTIGATION

2.1 What are the values of

(a) $(5 - (4 - (3 - (2 - 1)))) - (1 - (2 - (3 - (4 - 5))))$ and

(b) $(6 - (5 - (4 - (3 - (2 - 1))))) - (1 - (2 - (3 - (4 - (5 - 6)))))$?

2.2 Generalize the results of Exercise 2.1.

3. Used in measuring the width of a wire, one mil is equal to one thousandth of an inch. An inch is about 2.5 cm.

Which of these is approximately equal to one mil?

A $\frac{1}{40}$ mm B $\frac{1}{25}$ mm C $\frac{1}{4}$ mm D 25 mm E 40 mm

SOLUTION **A**

One mil is equal to one thousandth of an inch. An inch is about 2.5 cm, which is the same as 25 mm. Therefore one mil is approximately $\frac{25}{1000}$ mm, which is the same as $\frac{1}{40}$ mm.

FOR INVESTIGATION

3.1 There are twelve inches in one foot, three feet in one yard, and 1760 yards in one mile. Approximately how many metres are there in one mile?

4. For how many positive integer values of n is $n^2 + 2n$ prime?

A 0 B 1 C 2 D 3 E more than 3

Solution **B**

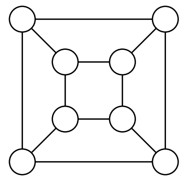
We have $n^2 + 2n = n(n + 2)$. Therefore $n^2 + 2n$ is divisible by n. Hence, for $n^2 + 2n$ to be prime, n can only have the value 1.

When $n = 1$, we have $n^2 + 2n = 3$, which is prime.

Therefore there is just one positive integer value of n for which $n^2 + 2n$ is prime.

5. Olive Green wishes to colour all the circles in the diagram so that, for each circle, there is exactly one circle of the same colour joined to it.

What is the smallest number of colours that Olive needs to complete this task?

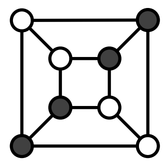


A 1 B 2 C 3 D 4 E 5

Solution **B**

If Olive uses just one colour, then each circle would be joined to three circles with the same colour as it. So one colour is not enough.

However, as the diagram shows, it is possible using just two colours to colour the circles so that each white circle is joined to just one white circle, and each black circle is joined to just one black circle.

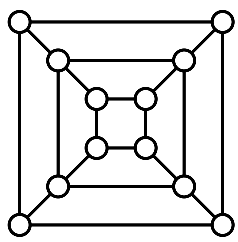


Therefore the smallest number of circles that Olive needs is two.

For investigation

5.1 Cherry Red wishes to colour all the circles in the diagram so that for each circle, there is exactly one circle with the same colour joined to it.

What is the smallest number of colours that Cherry needs to complete this task?



6. Each of the factors of 100 is to be placed in a 3 by 3 grid, one per cell, in such a way that the products of the three numbers in each row, column and diagonal are all equal. The positions of the numbers 1, 2, 50 and x are shown in the diagram.

x	1	50
2		

What is the value of x?

A 4 B 5 C 10 D 20 E 25

Solution **D**

Let P be the common product of the three numbers in each row. Then $P \times P \times P$ is the product of all the numbers in all three rows. Therefore P^3 is the product of all the factors of 100.

Because $100 = 2^2 \times 5^2$, it has the nine factors 1, $2^1 = 2$, $2^2 = 4$, $5^1 = 5$, $2^1 5^1 = 10$, $2^2 5^1 = 20$, $5^2 = 25$, $2^1 5^2 = 50$ and $2^2 5^2 = 100$. The product of these factors is

$$1 \times 2^1 \times 2^2 \times 5^1 \times 2^1 5^1 \times 2^2 5^1 \times 2^1 5^2 \times 5^2 \times 2^2 5^2 = 2^{1+2+1+2+1+2} \times 5^{1+1+1+2+2+2}$$
$$= 2^9 \times 5^9 \ (= 1\,000\,000\,000).$$

It follows that

$$P^3 = 2^9 \times 5^9$$

and therefore $P = 2^3 \times 5^3 = 1000$.

From the first row we have

$$x \times 1 \times 50 = 1000$$

and hence $x = 20$.

Note

In the context of the SMC it is not necessary to check that it is possible to complete the grid with all the factors of 100, so as to meet the condition that the product of the three numbers in each row, column and diagonal are all equal. However, you are asked to do this in Exercise 6.1.

Note, also that our solution does not use the position of the factor 2. Exercise 6.1 asks you to show that with 2 in the bottom left-hand cell, there is just one way to complete the grid.

For investigation

6.1 Show that it is possible to complete the grid with the factors of 100 so as to meet the required condition in just one way.

6.2 How many ways are there to complete the grid if the factor 2 does not have to be in the bottom left-hand cell?

6.3 Suppose that $n = p^2 q^2$, where p and q are different primes. Explain why n has nine factors.

6.4 Suppose that $n = p^a q^b$, where p and q are different primes and a and b are non-negative integers. How many factors does n have?

6.5 Find a general formula for the number of factors of a positive integer in terms of the exponents that occur in its prime factorization.

7. Lucy is asked to choose p, q, r and s to be the numbers 1, 2, 3 and 4, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

A $\frac{7}{12}$ B $\frac{2}{3}$ C $\frac{3}{4}$ D $\frac{5}{6}$ E $\frac{11}{12}$

SOLUTION **D**

To make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible, q and s need to be as large as possible, and so have the values 3 and 4.

Therefore the expression with the smallest value that Lucy can achieve is either $\frac{1}{3} + \frac{2}{4}$ or $\frac{2}{3} + \frac{1}{4}$.

Now,

$$\begin{aligned}\frac{1}{3} + \frac{2}{4} &= \frac{1}{3} + \frac{1}{4} + \frac{1}{4} \\ &< \frac{1}{3} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{2}{3} + \frac{1}{4}.\end{aligned}$$

It follows that the smallest value that Lucy can achieve is

$$\frac{1}{3} + \frac{2}{4} = \frac{4+6}{12} = \frac{10}{12} = \frac{5}{6}.$$

FOR INVESTIGATION

7.1 Lucy is asked to choose p, q, r, s, t and u to be the numbers 1, 2, 3, 4, 5 and 6, in some order, so as to make the value of $\frac{p}{q} + \frac{r}{s} + \frac{t}{u}$ as small as possible.

What is the smallest value Lucy can achieve in this way?

7.2 Explain why the statement in the first line of the above solution that

"to make the value of $\frac{p}{q} + \frac{r}{s}$ as small as possible, q and s need to be as large as possible"

is correct.

8. The number x is the solution to the equation $3^{(3^x)} = 333$.

Which of the following is true?

A $0 < x < 1$ B $1 < x < 2$ C $2 < x < 3$ D $3 < x < 4$ E $4 < x < 5$

SOLUTION **B**

The first few powers of 3 are

$$3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243 \text{ and } 3^6 = 729.$$

Because $3^{(3^x)} = 333$, it follows that

$$3^5 < 3^{(3^x)} < 3^6$$

and hence

$$5 < 3^x < 6.$$

Therefore

$$3^1 < 3^x < 3^2$$

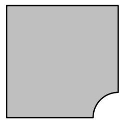
and hence

$$1 < x < 2.$$

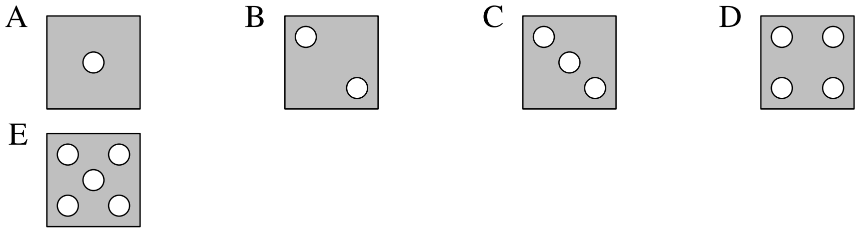
FOR INVESTIGATION

8.1 The number x is the solution to the equation $2^{2^x} = 10^6$. Find the integer n such that $n < x < n + 1$.

9. A square of paper is folded in half four times to obtain a smaller square. Then a corner is removed as shown.



Which of the following could be the paper after it is unfolded?



Solution **D**

Each time the square piece of paper is folded in half the number of layers of paper doubles. Therefore, after it has been folded four times, there are 16 layers.

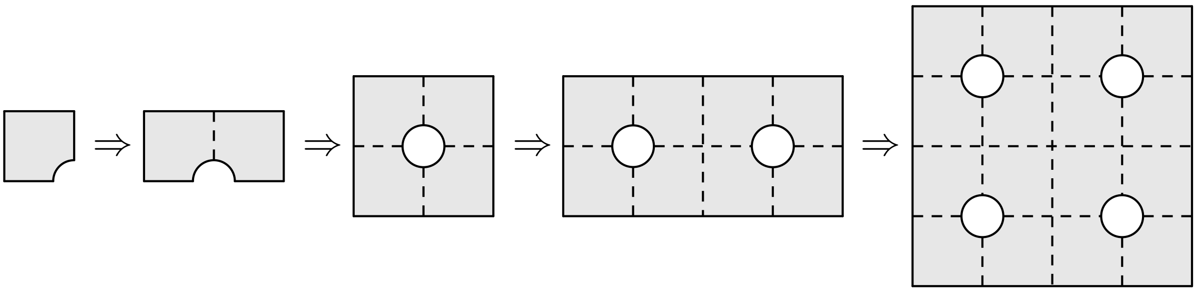
It follows that when the corner is then removed, altogether 16 quarter circles have been removed.

Hence, if these quarter circles come together to make complete circles after the paper has been unfolded, they will make four complete circles.

It follows that of the options given in the question, option D is the only one that is possible.

Note

In the context of the SMC it is sufficient to show that option D is the only possibility. For a complete answer it is also necessary to show that the pattern of option D can be achieved. This is shown by the diagram below. This shows that, provided none of the quarter-circles that are removed comes from the edge of paper, the paper will unfold to make the pattern of option D.



For investigation

9.1 What are the other possibilities for the paper after it has been unfolded?

10. Which of the following five values of n is a counterexample to the statement in the box below?

For a positive integer n, at least one of $6n - 1$ and $6n + 1$ is prime.

A 10 B 19 C 20 D 21 E 30

Solution **C**

A counterexample to the statement in the box is a value of n for which it is not true that at least one of $6n - 1$ and $6n + 1$ is prime. That is, it is a value of n for which neither $6n - 1$ nor $6n + 1$ is prime.

We set out the values of $6n - 1$ and $6n + 1$ for $n = 10,\ 19,\ 20,\ 21$ and 30 in the following table.

n	$6n - 1$	$6n + 1$
10	59	61
19	113	**115**
20	**119**	**121**
21	**165**	167
30	179	181

The values of $6n - 1$ and $6n + 1$ that are not prime are shown in bold.

We therefore see that for $n = 20$, neither $6n - 1$ nor $6n + 1$ is prime. Therefore $n = 20$ provides the required counterexample.

For investigation

10.1 Check that, of the numbers that occur in the table above, 59, 61, 113, 167, 179 and 181 are prime, and that 115, 119, 121 and 165 are not prime.

10.2 Show that if p is a prime number other than 2 or 3, then there is a positive integer n such that p is either equal to $6n - 1$ or $6n + 1$.

11. For how many integer values of k is $\sqrt{200 - \sqrt{k}}$ also an integer?

A 11 B 13 C 15 D 17 E 20

Solution **C**

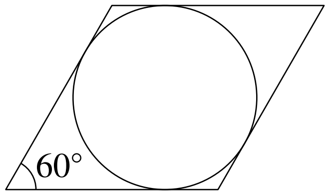
The number $\sqrt{200 - \sqrt{k}}$ is an integer if, and only if, the number $200 - \sqrt{k}$ is a square.

Now $0 \le 200 - \sqrt{k} \le 200$. There are 15 squares in this range, namely n^2 for integer values of n with $0 \le n \le 14$.

We have that $200 - \sqrt{k} = n^2$, if, and only if $k = (200 - n^2)^2$.

Hence there are 15 integer values of k for which $\sqrt{200 - \sqrt{k}}$ is an integer, namely, $k = (200 - n^2)^2$ for $0 \le n \le 14$.

12. A circle with radius 1 touches the sides of a rhombus, as shown. Each of the smaller angles between the sides of the rhombus is 60°.

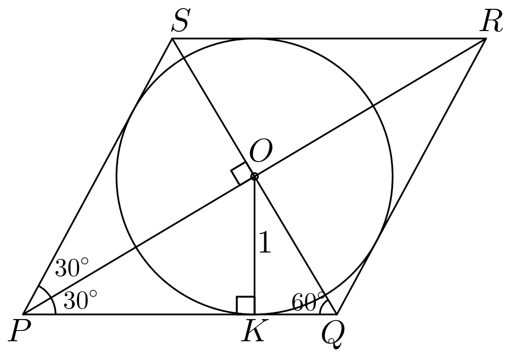


What is the area of the rhombus?

A 6 B 4 C $2\sqrt{3}$ D $3\sqrt{3}$ E $\dfrac{8\sqrt{3}}{3}$

Solution **E**

Let P, Q, R and S be the vertices of the rhombus.



We leave it to the reader to prove that the diagonals of a rhombus bisect the angles of the rhombus and meet at right angles (see Exercise 12.1).

Let O be the point where the diagonals PR and QS of the rectangle meet.

We also leave it to the reader to prove that the four triangles POQ, QOR, ROS and SOP are congruent (see Exercise 12.1) and that O is the centre of the circle (see Exercise 12.2).

We let K be the point where PQ touches the circle. Then $OK = 1$. Because the radius of a circle is at right angles to the tangent at the point where the radius meets the circle, $\angle PKO = 90°$.

From the right-angled triangle PKO we have

$$\frac{OK}{OP} = \sin 30° = \frac{1}{2},$$

and therefore, since $OK = 1$, it follows that $OP = 2$.

Because $\angle POQ = 90°$, it follows from the triangle POQ that $\angle OQK = 60°$, and hence

$$\frac{OK}{OQ} = \sin 60° = \frac{\sqrt{3}}{2},$$

and therefore $OQ = \dfrac{2}{\sqrt{3}}$.

It now follows that the area of the triangle POQ is given by

$$\frac{1}{2}(OP \times OQ) = \frac{1}{2}\left(2 \times \frac{2}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}.$$

Therefore, as the rhombus is made up of four triangles each congruent to the triangle POQ, the area of the rhombus is

$$4 \times \frac{2}{\sqrt{3}} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}.$$

Note: See Exercise 12.4 for a way to remember the values of sin(30°) and sin(60°).

For investigation

12.1 (a) Prove that the diagonals of a rhombus divide the rhombus into four congruent triangles.

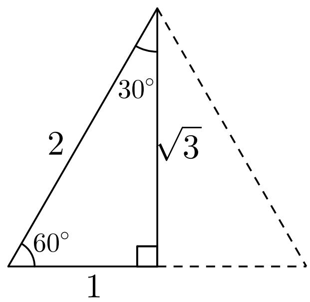
(b) Deduce that the diagonals of a rhombus bisect the angles of the rhombus and meet at right angles.

12.2 Prove that the diagonals of a rhombus meet at the centre of a circle that touches the four sides of the rhombus

12.3 Prove that a radius of a circle is at right angles to the tangent at the point where the radius meets the circle.

12.4 A triangle with angles of 30°, 60° and 90° forms one half of an equilateral triangle, as shown in the diagram. We suppose that the side length of the equilateral triangle is 2.

Then the shortest side of the $30°, 60°, 90°$ triangle is 1.



(a) Use Pythagoras' Theorem to check that the third side of this triangle has length $\sqrt{3}$.

(b) Use this triangle to check that that $\sin(30°) = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$.

(c) Use this triangle to find the values of $\cos(30°)$, $\tan(30°)$, $\cos(60°)$ and $\tan(60°)$.

12.5 Use a different triangle to obtain the values of $\sin(45°)$, $\cos(45°)$ and $\tan(45°)$.

13. Anish has a number of small congruent square tiles to use in a mosaic. When he forms the tiles into a square of side n, he has 64 tiles left over. When he tries to form the tiles into a square of side $n + 1$, he has 25 too few.

How many tiles does Anish have?

A 89 B 1935 C 1980 D 2000 E 2019

Solution **D**

Because Anish has 64 tiles left over when he forms a square of side n, he has $n^2 + 64$ tiles.

Because Anish has 25 tiles too few to make a square of side $n + 1$, he has $(n + 1)^2 - 25$ tiles.

Therefore $n^2 + 64 = (n + 1)^2 - 25$. Now

$$\begin{aligned} n^2 + 64 = (n + 1)^2 - 25 &\Leftrightarrow n^2 + 64 = (n^2 + 2n + 1) - 25 \\ &\Leftrightarrow 2n = 64 + 25 - 1 \\ &\Leftrightarrow 2n = 88 \\ &\Leftrightarrow n = 44. \end{aligned}$$

Because $n = 44$, the number of tiles that Anish has is given by $44^2 + 64 = 1936 + 64 = 2000$.

For investigation

13.1 For $n = 44$, check that $(n + 1)^2 - 25$ also equals 2000.

13.2 Anish has exactly enough square 1×1 tiles to form a square of side m. He would need 2019 more tiles to form a square of side $m + 1$.

How many tiles does Anish have?

13.3 Anish has exactly enough $1 \times 1 \times 1$ cubes to form a cube of side m. He would need 397 more cubes to form a cube of side $m + 1$.

How many cubes does Anish have?

14. One of the following is the largest square that is a factor of 10!. Which one?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$

A $(4!)^2$ B $(5!)^2$ C $(6!)^2$ D $(7!)^2$ E $(8!)^2$

Solution **C**

Commentary

If you are familiar with the values of $n!$ for small values of n, it is not difficult to spot the answer quite quickly, as in Method 1. If not, a more systematic method is to work out the prime factorization of 10!, as in Method 2.

Method 1

We have

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= (1 \times 2 \times 3 \times 4 \times 5 \times 6) \times 7 \times (8 \times 9 \times 10) \\ &= 6! \times 7 \times (8 \times 9 \times 10) \\ &= 6! \times 7 \times 720 \\ &= 6! \times 7 \times 6! \\ &= (6!)^2 \times 7. \end{aligned}$$

It follows that $(6!)^2$ is the largest square that is a factor of 10!.

Method 2

We have

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= 1 \times 2 \times 3 \times 2^2 \times 5 \times (2 \times 3) \times 7 \times 2^3 \times 3^2 \times (2 \times 5) \\ &= 2^8 \times 3^4 \times 5^2 \times 7 \\ &= (2^4 \times 3^2 \times 5)^2 \times 7. \end{aligned}$$

It follows that $2^4 \times 3^2 \times 5$ is the largest square that is a factor of 10!. Now $2^4 \times 3^2 \times 5 = 720 = 6!$ and therefore $(6!)^2$ is the largest square that is a factor of 10!.

For investigation

14.1 Which is the largest square that is a factor of (a) 12!, and (b) 14! ?

14.2 List the values of $n!$ for all positive integers n with $n \leq 10$.

15. The highest common factors of all the pairs chosen from the positive integers Q, R and S are three different primes.

What is the smallest possible value of $Q + R + S$?

A 41 B 31 C 30 D 21 E 10

Solution **B**

We use the notation $\text{HCF}(X, Y)$ for the highest common factor of the two integers X and Y.

Suppose that $\text{HCF}(Q, R) = a$, $\text{HCF}(Q, S) = b$ and $\text{HCF}(R, S) = c$, where a, b and c are three different primes.

It follows that both a and b are factors of Q. Therefore the smallest possible value of Q is ab. Likewise, the smallest possible values of R and S are ac and bc, respectively.

We seek the smallest possible value of $Q + R + S$, that is, of $ab + ac + bc$, where a, b and c are different primes. To do this we choose the values of a, b and c to be the three smallest primes, that is 2, 3 and 5, in some order.

Because $ab + ac + bc$ is symmetric in a, b and c, the order does not matter. With $a = 2$, $b = 3$ and $c = 5$, we have

$$ab + ac + bc = 2 \times 3 + 2 \times 5 + 3 \times 5 = 6 + 10 + 15 = 31.$$

We deduce that the smallest possible value of $Q + R + S$ is 31.

For investigation

15.1 Show that if, in the above solution, a, b and c are given the values 2, 3 and 5 in some other order, then the value of $ab + ac + bc$ is again 31.

15.2 Show that if, in the above solution, any of 2, 3 and 5 is replaced by a prime larger than 5, then the resulting value of $Q + R + S$ is greater thsn 31.

15.3 The highest common factors of all the pairs chosen from the positive integers Q, R, S and T are six different primes.

What is the smallest possible value of $Q + R + S + T$?

16. The numbers x, y and z satisfy the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$.

What is the mean of x, y and z?

A 10 B 11 C 12 D 13 E 14

Solution **A**

Commentary

The mean of x, y and z is $\frac{1}{3}(x + y + z)$. Therefore to answer this question we need to find the value of $x + y + z$. We are given just two equations for the three unknowns x, y and z. It follows that if these equations have a solution, they will have an infinite number of solutions.

A systematic method for answering the question would be to use the two equations to find expressions for two of the unknowns in terms of the third unknown. For example, we could find x and y in terms of z, and thus work out $x + y + z$ in terms of z.

However, the wording of the question suggests that $x + y + z$ is independent of z. Thus a good starting point is to try to find a way to use the two equations we are given to find a value for $x + y + z$ without the need to find x and y in terms of z.

We are given that

$$9x + 3y - 5z = -4 \qquad (1)$$

and

$$5x + 2y - 2z = 13 \qquad (2)$$

If we multiply equation (2) by 2, and subtract equation (1) we obtain

$$2(5x + 2y - 2z) - (9x + 3y - 5z) = 2(13) - (-4),$$

that is,

$$10x + 4y - 4z - 9x - 3y + 5z = 26 + 4,$$

that is,

$$x + y + z = 30.$$

We deduce that $\frac{1}{3}(x + y + z) = 10$.

Hence the mean of x, y and z is 10.

For investigation

16.1 (a) Use the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$ to find expressions for y and z in terms of x.

(b) Use your answers to part (a) to show that for all values of x, we have $x + y + z = 30$.

16.2 The numbers x, y and z satisfy the equations $3x - 5y + 7z = 4$ and $4x - 8y + 10z = 6$. What is the mean of x, y and z?

16.3 You may know that the equations $9x + 3y - 5z = -4$ and $5x + 2y - 2z = 13$ represent two-dimensional planes in three-dimensional space. What can you deduce about these planes from the fact that the equations imply that $x + y + z = 30$?

17. Jeroen writes a list of 2019 consecutive integers. The sum of his integers is 2019.

What is the product of all the integers in Jeroen's list?

A 2019^2 B $\frac{2019 \times 2020}{2}$ C 2^{2019} D 2019 E 0

SOLUTION **E**

In a list of 2019 consecutive *positive* integers, at least one of them will be greater than or equal to 2019, and therefore the sum of these integers will be greater than 2019. So the integers in Jeroen's list are not all positive.

The sum of 2019 *negative* integers is negative and therefore cannot be equal to 2019. So the integers in Jeroen's list are not all negative.

We deduce that Jeroen's list of consecutive integers includes both negative and positive integers.

Because the integers in the list are consecutive it follows that one of them is 0.

Therefore the product of all the numbers in Jeroen's list is 0.

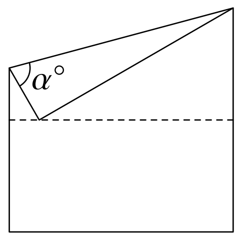
FOR INVESTIGATION

17.1 Note that we were able to answer this question without finding a list of 2019 consecutive integers with sum 2019. Show that there is just one list of 2019 consecutive integers whose sum is 2019, and find it.

17.2 Find all the lists of at least two consecutive integers whose sum is 2019.

17.3 Investigate which positive integers can be expressed as the sum of two or more consecutive positive integers.

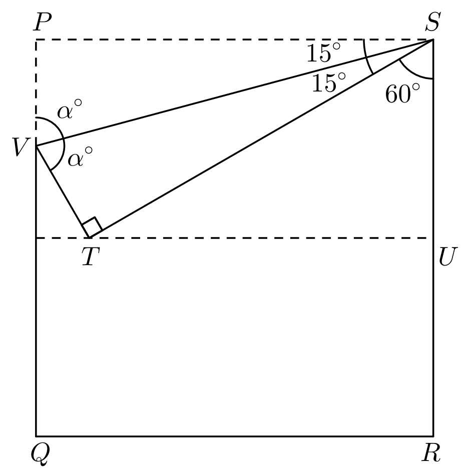
18. Alison folds a square piece of paper in half along the dashed line shown in the diagram. After opening the paper out again, she then folds one of the corners onto the dashed line.



What is the value of α?

A 45 B 60 C 65 D 70 E 75

SOLUTION **E**



Let the vertices of the square be P, Q, R and S, as shown. Let T be the position to which P is folded, and let U and V be the points shown in the diagram.

Because after the fold the triangle PSV coincides with the triangle TSV, these triangles are congruent. In particular $TS = PS$ and $\angle PSV = \angle TSV$.

Therefore SUT is a triangle with a right angle at U and in which $SU = \frac{1}{2}SR = \frac{1}{2}PS = \frac{1}{2}TS$. It follows that [see Exercise 12.4] $\angle TSU = 60°$.

Because $\angle USP = 90°$ and $\angle PSV = \angle TSV$, it follows that $\angle TSV = \frac{1}{2}(90 - 60)° = 15°$.

Therefore, because the sum of the angles in the triangle TSV is 180°, we have

$$\alpha + 15 + 90 = 180,$$

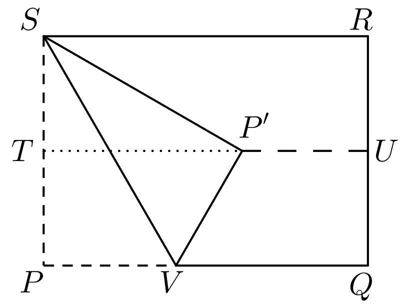
and therefore $\alpha = 75$.

FOR INVESTIGATION

18.1 Suppose that $PQRS$ is a rectangular piece of paper, in which PQ is longer than QR.

The paper is folded in half along the line TU, and then unfolded.

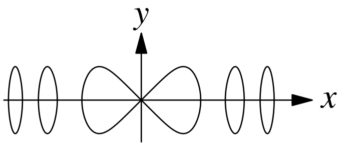
Next, the paper is folded along the line SV through S so that the corner P ends up at the point P' on the first fold line TU.

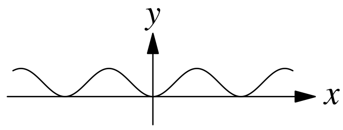


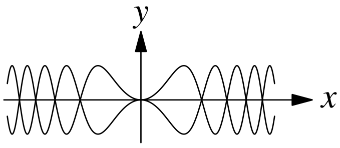
(a) Prove that $\angle SVP' = 60°$.

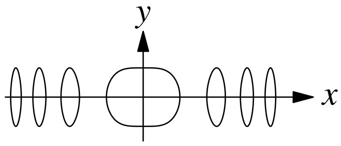
(b) Show how a rectangular piece of paper $PQRS$ may be folded to make an equilateral triangle, provided that the ratio $PQ : QR$ is sufficiently large.

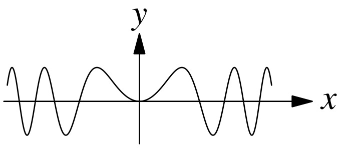
19. Which of the following could be the graph of $y^2 = \sin(x^2)$?

A


B


C


D


E


SOLUTION **A**

We have $\sin(0^2) = \sin(0) = 0$. The equation $y^2 = 0$ has only one solution, namely $y = 0$. Therefore the point $(0, 0)$ lies on the graph of $y^2 = \sin(x^2)$, and there is no point on the graph of the form $(0, b)$ with $b \neq 0$. This rules out option D.

If the point with coordinates (a, b) is on the graph, $b^2 = \sin(a^2)$. Hence we also have $(-b)^2 = \sin(a^2)$. Therefore the point with coordinates $(a, -b)$ also lies on the graph.

In other words, the graph of $y^2 = \sin(x^2)$ is symmetric about the x-axis. This rules out the graphs of options B and E.

There are positive values of x for which $\sin(x) < 0$. Suppose, for example, $a > 0$ and $\sin(a) < 0$. Then $\sin((\sqrt{a})^2) < 0$ and so cannot be equal to y^2 for any real number y. That is, there is no value of y for which the point with coordinates $(\sqrt{a}, y)$ lies on the graph.

In other words, the graph is not defined for all values of x. This rules out option C. [Note that this argument could also be used to rule out options B and E.]

This leaves option A as the only possibility for the graph.

FOR INVESTIGATION

19.1 For each of the following equations decide which, if any, of the options in this question could be its graph.

(a) $y = \sin(x^2)$,

(b) $y = \cos(x^2)$,

(c) $y^2 = \cos(x^2)$,

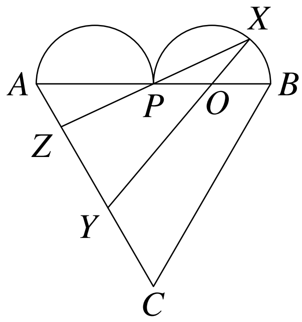
(d) $y = \sin^2(x)$,

(e) $y = \cos^2(x)$,

(f) $y^2 = \sin^2(x^2)$,

(g) $y^2 = \cos^2(x^2)$.

20. The "heart" shown in the diagram is formed from an equilateral triangle ABC and two congruent semicircles on AB. The two semicircles meet at the point P. The point O is the centre of one of the semicircles. On the semicircle with centre O, lies a point X. The lines XO and XP are extended to meet AC at Y and Z respectively. The lines XY and XZ are of equal length.



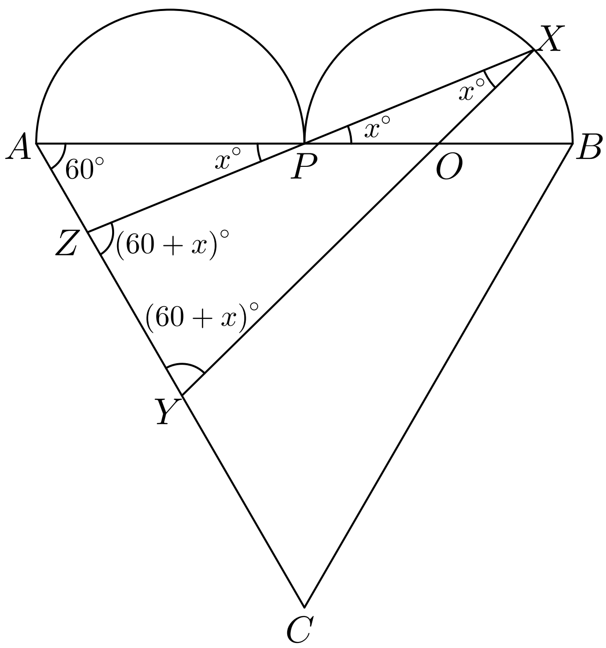
What is $\angle ZXY$?

A 20° B 25° C 30° D 40° E 45°

Solution **A**

Let $\angle ZXY = x°$.

The triangle POX is isosceles because OP and OX are radii of a semicircle and hence are equal. Therefore, $\angle OPX = \angle OXP = x°$.



Since $\angle APZ$ and $\angle OPX$ are vertically opposite, $\angle APZ = \angle OPX = x°$.

Because it is an angle of an equilateral triangle, $\angle ZAP = 60°$.

It now follows from the External Angle Theorem applied to the triangle AZP that $\angle YZX = \angle ZAP + \angle APZ = (60 + x)°$.

Because $XY = XZ$, the triangle XYZ is isosceles and therefore $\angle ZYX = \angle YZX = (60 + x)°$.

We now apply the fact that the sum of the angles of a triangle is 180° to the triangle XYZ. This gives

$$x + (60 + x) + (60 + x) = 180.$$

Hence

$$3x + 120 = 180.$$

It follows that $x = 20$ and hence $\angle ZXY = 20°$.

For investigation

20.1 Prove that if, in the diagram of this question, the triangle ABC is not necessarily equilateral, but is isosceles, with $CB = CA$, and with the lines XY and XZ again of equal length, then $\angle ZXY = \frac{1}{3}\angle ACB$.

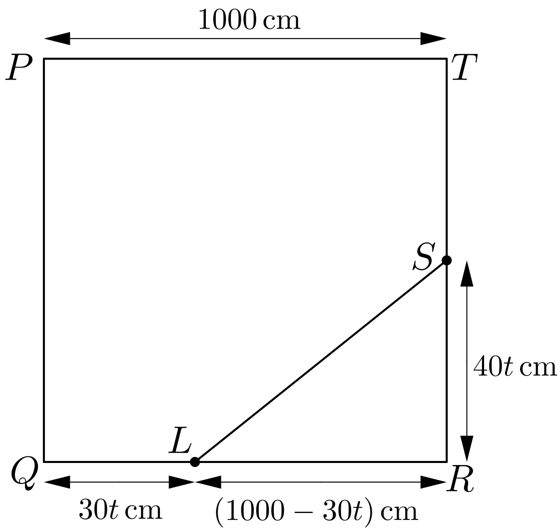
21. In a square garden $PQRT$ of side 10 m, a ladybird sets off from Q and moves along edge QR at 30 cm per minute. At the same time, a spider sets off from R and moves along edge RT at 40 cm per minute.

What will be the shortest distance between them, in metres?

A 5 B 6 C $5\sqrt{2}$ D 8 E 10

Solution **D**

Let L be the point that the ladybird reaches after t minutes. Then $QL = 30t$ cm. The side length of the square is 10 m which is the same as 1000 cm. Therefore the length of LR is $(1000 - 30t)$ cm.



Let S be the point that the spider reaches after t minutes. Then $RS = 40t$ cm.

Let the length of LS be x cm. This is the distance between the ladybird and the spider after t minutes.

Because $PQRT$ is a square, $\angle LRS = 90°$. By Pythagoras' Theorem applied to the triangle LRS, we have

$$\begin{aligned} x^2 &= (1000 - 30t)^2 + (40t)^2 \\ &= 1\,000\,000 - 60\,000t + 900t^2 + 1600t^2 \\ &= 2500t^2 - 60\,000t + 1\,000\,000 \\ &= 2500(t^2 - 24t + 400) \\ &= 2500\big((t - 12)^2 + 256\big). \end{aligned}$$

Because $(t - 12)^2 \geq 0$ for all values of t, it follows that $x^2 \geq 2500 \times 256$ for all values of t. Since $x^2 = 2500 \times 256$ when $t = 12$, it follows that the smallest value of x^2 is 2500×256. Therefore the smallest value taken by x is $\sqrt{2500 \times 256}$, which is equal to 50×16, that is, 800. Hence the shortest distance between the ladybird and the spider is 800 cm which is 8 m.

For investigation

21.1 In the above solution we found the smallest value of the quadratic $t^2 - 24t + 400$ by the method of *completing the square*.

This works for quadratics but not for other functions. The *differential calculus* gives us a general method for finding maximum and minimum values of functions. If you have already met calculus, use it to find the minimum value of $t^2 - 24t + 400$.

21.2 Sketch the graph of $y = t^2 - 24t + 400$.

21.3 Find the minimum value of $2t^2 + 6t - 12$.

22. A function f satisfies the equation $(n-2019)f(n)-f(2019-n)=2019$ for every integer n.

What is the value of $f(2019)$?

A 0 B 1 C 2018×2019 D 2019^2 E 2019×2020

SOLUTION **C**

Putting $n=0$ in the equation

$$(n-2019)f(n)-f(2019-n)=2019 \qquad (1)$$

gives

$$-2019f(0)-f(2019)=2019 \qquad (2)$$

from which it follows that

$$f(2019)=-2019f(0)-2019. \qquad (3)$$

Putting $n=2019$ in equation (1) gives

$$-f(0)=2019 \qquad (4)$$

and hence

$$f(0)=-2019. \qquad (5)$$

Substituting from (5) in (3) gives

$$\begin{aligned} f(2019) &= -2019\times-2019-2019 \\ &= 2019\times2019-2019 \\ &= 2019(2019-1) \\ &= 2019\times2018. \end{aligned}$$

Therefore, the value of $f(2019)$ is 2018×2019.

FOR INVESTIGATION

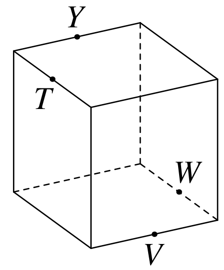
22.1 What is the value of $f(1)$?

22.2 Find the general formula for $f(n)$ in terms of n.

23. The edge-length of the solid cube shown is 2. A single plane cut goes through the points Y, T, V and W which are midpoints of the edges of the cube, as shown.

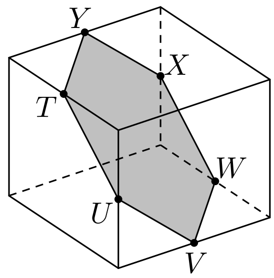
What is the area of the cross-section?

A $\sqrt{3}$ B $3\sqrt{3}$ C 6 D $6\sqrt{2}$ E 8



SOLUTION **B**

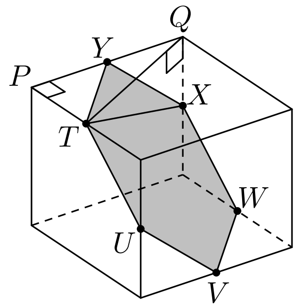
The plane which goes through the points Y, T, V and W also meets the edges of the cube at the points U and X, as shown.



We leave it as an exercise to check that the points U and X are also the midpoints of the edges on which they lie.

The relevant cross-section is therefore the hexagon $TUVWXY$. We first show that this is a regular hexagon.

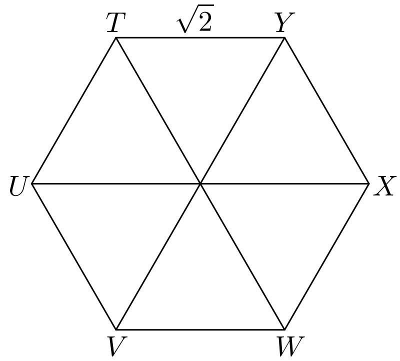
Let P and Q be the vertices of the cube, as shown. Applying Pythagoras' Theorem to the right-angled triangle TPY, gives $TY^2 = PT^2 + PY^2 = 1^2 + 1^2 = 2$. Therefore $TY = \sqrt{2}$. In a similar way it follows that each edge of the hexagon $TUVWXY$ has length $\sqrt{2}$.



Applying Pythagoras' Theorem to the right-angled triangle TPQ gives $QT^2 = PT^2 + PQ^2 = 1^2 + 2^2 = 5$. The triangle TXQ has a right angle at Q because the top face of the cube is perpendicular to the edge through Q and X. Hence, by Pythagoras' Theorem, $TX^2 = QT^2 + QX^2 = 5 + 1^2 = 6$. Hence $TX = \sqrt{6}$.

It follows that in the triangle TXY we have $TY = YX = \sqrt{2}$ and $TX = \sqrt{6}$. We leave it as an exercise to check that it follows that $\angle TYX = 120°$. It follows similarly that each angle of $TUVWXY$ is $120°$. Hence this hexagon is regular.

The regular hexagon $TUVWXY$ may be divided into six congruent equilateral triangles each with side length $\sqrt{2}$.



We leave it as an exercise to check that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. It follows that equilateral triangles with side lengths $\sqrt{2}$ have area $\frac{\sqrt{3}}{4}(\sqrt{2})^2 = \frac{\sqrt{3}}{2}$.

Therefore the area of the cross-section is $6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$.

Note: The facts you are asked to check in the above solution are given in as Exercises 23.1, 23.2 and 23.3 below.

For investigation

23.1 Show that a triangle with side lengths $\sqrt{6}$, $\sqrt{2}$ and $\sqrt{2}$, has a largest angle of 120° and two angles each of 30°.

23.2 Show that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$.

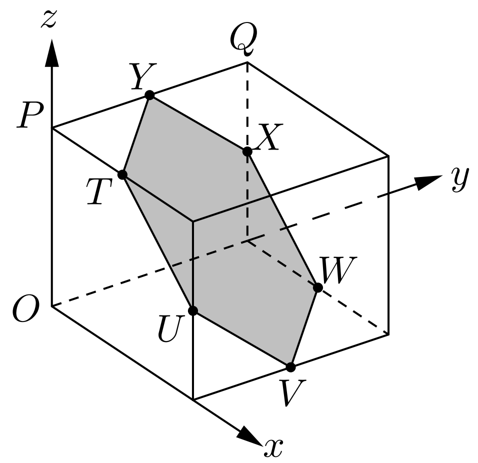
23.3 Let O be the bottom left-hand vertex of the cube as shown.

We let O be the origin of a system of three-dimensional coordinates with $x-$axis, $y-$axis and $z-$axis as shown.

The coordinates in this coordinate system of the point T are $(1, 0, 2)$.

(a) Find the coordinates in this coordinate system of the points Y and V.

(b) The equation of a plane in three-dimensions has the form $ax + by + cz = d$, where a, b, c and d are constants.



Find the equation of the plane that goes through all the points T, Y and V.

(c) Hence, verify that the plane that goes through the points T, Y and V also goes through the point W.

(d) Use the equation of the plane through the points T, Y and V to find the coordinates of the points U and X. Deduce that U and X are the midpoints of the edges on which they lie.

23.4 Why does it not follow just from the fact that each edge of hexagon $TUVWXY$ has length $\sqrt{2}$ that the hexagon is regular?

24. The numbers x, y and z are given by $x = \sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}$, $y = \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}}$ and $z = \sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$.

What is the value of xyz ?

A 1 B –6 C –8 D 18 E 12

Solution **E**

Commentary

A natural first thought on reading this question is that working out the product xyz, without the use of a calculator, will be horribly complicated, and that, given the time constraints of the SMC, it would be sensible to skip this question.

But then you might realize that, because this is an SMC question, there could well be a smarter way to tackle this question.

Because the expressions for x, y and z involve lots of square roots, a smart method might be to work out x^2, y^2 and z^2 separately and hence find $x^2y^2z^2$, which is equal to $(xyz)^2$. It is, perhaps, surprising that this approach works out well.

We have

$$\begin{aligned}
x^2 &= \left(\sqrt{12 - 3\sqrt{7}} - \sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(\sqrt{12 - 3\sqrt{7}}\right)^2 - 2\left(\sqrt{12 - 3\sqrt{7}}\right)\left(\sqrt{12 + 3\sqrt{7}}\right) + \left(\sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(\sqrt{12 - 3\sqrt{7}}\right)^2 - 2\sqrt{\left((12 - 3\sqrt{7})(12 + 3\sqrt{7})\right)} + \left(\sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(\sqrt{12 - 3\sqrt{7}}\right)^2 - 2\left(\sqrt{12^2 - (3\sqrt{7})^2}\right) + \left(\sqrt{12 + 3\sqrt{7}}\right)^2 \\
&= \left(12 - 3\sqrt{7}\right) - 2\sqrt{144 - 63} + \left(12 + 3\sqrt{7}\right) \\
&= \left(12 - 3\sqrt{7}\right) - 2\sqrt{81} + \left(12 + 3\sqrt{7}\right) \\
&= 24 - 2 \times 9 \\
&= 6.
\end{aligned}$$

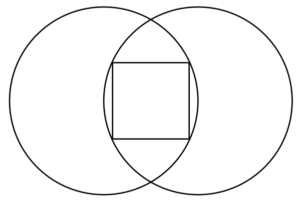
We leave it as an exercise to check that, similarly, $y^2 = 12$ and $z^2 = 2$. It follows that $(xyz)^2 = x^2y^2z^2 = 6 \times 12 \times 2 = 144$. Hence xyz equals either 12 or -12.

Because $12 - 3\sqrt{7} < 12 + 3\sqrt{7}$, we see that $x < 0$. Similarly $y < 0$ and $z > 0$. Hence $xyz > 0$. We conclude that $xyz = 12$.

For investigation

24.1 Check that $y^2 = 12$ and that $z^2 = 2$.

25. Two circles of radius 1 are such that the centre of each circle lies on the other circle. A square is inscribed in the space between the circles.



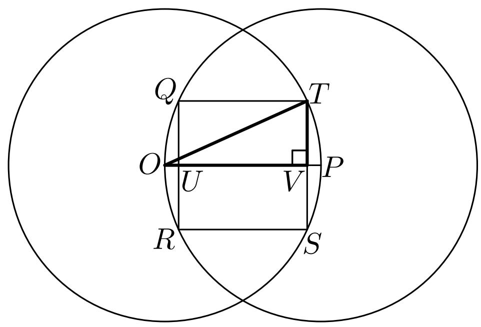
What is the area of the square?

A $2 - \dfrac{\sqrt{7}}{2}$ B $2 + \dfrac{\sqrt{7}}{2}$ C $4 - \sqrt{5}$

D 1 E $\dfrac{\sqrt{5}}{5}$

Solution **A**

We let O and P be the centres of the circles, Q, R, S and T be the vertices of the square, and U and V be the points where the line OP meets the edges of the square, as shown in the diagram.



We let x be the side length of the square.

The point V is the midpoint of ST and therefore $VT = \frac{1}{2}x$.

Because OT and OP are radii of the circle with centre O, we have $OT = OP = 1$.

Because $OP = 1$ and $UV = QT = x$, we have $OU + VP = 1 - x$. Since $OU = VP$, it follows that $OU = \frac{1}{2}(1 - x)$. Therefore $OV = OU + UV = \frac{1}{2}(1 - x) + x = \frac{1}{2}(1 + x)$.

Therefore, applying Pythagoras' Theorem to the right-angled triangle OVT, we obtain $OV^2 + VT^2 = OT^2$. That is, $\left(\frac{1}{2}(1+x)\right)^2 + \left(\frac{1}{2}x\right)^2 = 1^2$. This equation expands to give $\frac{1}{4} + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{4}x^2 = 1$. The last equation may be rearranged to give $2x^2 + 2x - 3 = 0$.

Hence, by the formula for the roots of a quadratic equation, $x = \dfrac{-2 \pm \sqrt{4 + 24}}{4} = \frac{1}{2}(-1 \pm \sqrt{7})$. Because $x > 0$, we deduce that $x = \frac{1}{2}(-1 + \sqrt{7})$.

Therefore the area of the square is given by

$$x^2 = \left(\tfrac{1}{2}(-1 + \sqrt{7})\right)^2 = \tfrac{1}{4}(1 - 2\sqrt{7} + 7) = \tfrac{1}{4}(8 - 2\sqrt{7}) = 2 - \frac{\sqrt{7}}{2}.$$

For investigation

25.1 Prove that the following claims made in the above solution are correct.

(a) The point V is the midpoint of ST.

(b) $OU = VP$.

(c) $\angle OVT = 90°$.